# Portfolio Selection Problems

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# Outline

CAPM

Tests of CAPM

Arbitrage Pricing Theory

Multi-stage models

Continuous-time models

Arbitrage Pricing Theory

New approaches reacting to drawbacks

# Diversification

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$$\mathrm{var}(r_1/2 + r_2/2) \leq \mathrm{var}(r_1) = \mathrm{var}(r_2)$$

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Markowitz model:

$$\boxed{\max_{\mathbf{1}'\pi=1,\pi\geq 0} \mathbb{E}(r'\pi) - \lambda \mathrm{var}(r'\pi)}$$

$r \in \mathbb{R}^N$ - stochastic asset returns
(i.e. a quadratic programming problem).

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## Theorem (von Neumann, Morgenstern)

*Given some (reasonable) requirements, $\preceq$ may be represented by an (utility) function $u$, i.e.*

$$x \preceq y \Leftrightarrow \mathbb{E}u(x) \leq \mathbb{E}u(y)$$

*for all pairs of r.v.'s $x, y$.*

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Too general to solve.

On the other hand, the solution of

$$\text{(MV)} \qquad \boxed{\min_{\mathbf{1}'\pi=1, \mathbb{E}(r'\pi)=\mu} \operatorname{var}(r'\pi)}$$

is closed form and unique for each $\mu \in \mathbb{R}$ given that $\operatorname{var}(r)$ is regular.

# Capital asset pricing model

[Sharpe, 60's - Nobel Prize 1990]

Assp. i. $r_0 \equiv r_f$ for some deterministic $r_f$ (risk free rate).

Assp. ii. $\mathbb{E}(r'\pi) \uparrow \Rightarrow \ \mathbb{E}u(r'\pi) \uparrow, \quad \mathrm{var}(r'\pi) \uparrow \Rightarrow \ \mathbb{E}u(r'\pi) \downarrow .$

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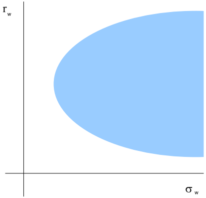


All portfolios of risky assets in the mean / st. dev. plane

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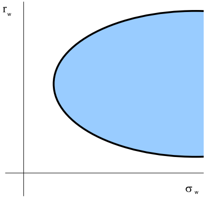
Solutions of [MV]



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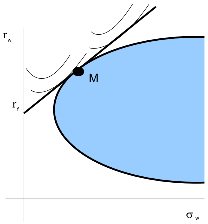


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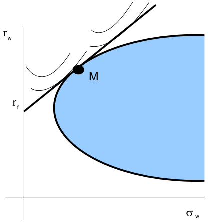


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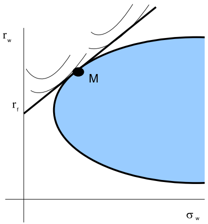


Linear combinations of the r.f. asset and $M$ (market portfolio) dominate all the other portfolios
$\Longrightarrow$ In equilibrium all investors hold the same proportion of risky assets (being their overall proportion).

# Remarks

- The market portfolio is tried to be mimicked by a stock index (e.g. PX50).
- After the emergence of CAPM, mutual funds “holding index” appeared.
- CAPM remains valid even without the possibility of short sales, there exist various modifications.

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# Some math

- Let $\mu > r_f$ be a prescribed rate of return. Problem (MV) is clearly equivalent to

$$\text{(MV')} \qquad \boxed{\min_{\pi \in \mathbb{R}^N, \mathbb{E}(r'\pi)+(1-\mathbf{1}'\pi)r_f=\mu} \pi' V \pi}$$

  where $r \in \mathbb{R}^N$ is the rate of return of the *risky* assets and $V$ their variance matrix.

- We may write

$$\pi^\mu = (1-\delta^\mu)\pi_M$$

$$\delta^\mu = 1 - \frac{(\mu - r_f)(B - Ar_f)}{Ar_f^2 - 2Br_f + C}, \pi_M = \frac{V^{-1}(\mathbb{E}r - r_f \cdot 1)}{B - Ar_f}.$$

- Because $\pi_M$ is a solution of (MV') for $\mu = \mu_M = (C - Br_f)/(B - Ar_f)$, it is one of the variance minimizing portfolios

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- Because $\pi_M$ is a solution of (MV') for $\mu = \mu_M = (C - Br_f)/(B - Ar_f)$, it is one of the variance minimizing portfolios $\Rightarrow$ Two funds separation!

# Risk premium expressed

After some math (Dupačová at al, I.10.1) we get

$$\underbrace{\mathbb{E}r - r_f \cdot 1}_{\text{risk premium}} = \frac{\rho_M}{\sigma_M^2}(\mu_M - r_f)$$

where

$$\sigma_M^2 = \mathrm{var}(\pi_M' r) = \frac{\mu_M - r_f}{B - A r_f},$$

$$\rho_M = \mathrm{cov}(r - \mathbf{1}' r_f, \pi_M r) = \frac{\mathbb{E}r - r_f \cdot 1}{B - A r_f}$$

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# A simple test of CAPM

The equation for the risk premium impies that, given $i \leq N$,

$$r_i - r_f = \beta_i(r_M - r_f) + \epsilon_i, \quad \mathbb{E}\epsilon_i = 0, cov(\epsilon_i, r_M) = 0$$

for some $\beta_i$ where $r_M$ is the return of the market portfolio.

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for some $\beta_i$ where $r_M$ is the return of the market portfolio.
For each $1 \leq \tau \leq T$, denote $r_f^\tau, r^\tau, r_M^\tau, \epsilon^\tau$ the values of respective quantities at time $\tau$.

For each $1 \leq i \leq N$, consider the model

$$r_i^\tau - r_f = \alpha_i + \beta_i(r_M^\tau - r_f^\tau) + \epsilon_i^\tau, \quad 1 \leq \tau \leq T$$

and test $H_0 : \alpha_i = 0$ (we take the values of stock index instead of unknown $r_M$ and an interest rate index - e.g. Pribor - as $r_f$). If the test is significant, it is an evidence against CAPM.

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Remark: $\beta_i$ is interpreted as a measure of the riskiness of the $i$-th stock.

# Homework

Each student should choose a single Czech stock and estimate the above model, check its quality, interpret its $\beta$, test whether $\alpha$ is zero and formulate implications of the test. The estimate should be based on at least 100 observations. The work is due by the midterm and should be sent to smid@utia.cas.cz.

# Arbitrage Pricing Theory (APT)

Assp. a. The agents have homogenous expectations (i.e. all agents assume the same expectations of involved random variables)

Assp. b $r_i = a_i + \sum_{i=1}^{k} b_{i,j} F_j + \epsilon_i$, $i = 1, \ldots, N$ where

- $F_1, \ldots, F_k$ are some factors
- $\mathbb{E}\epsilon_i = 0$, $\mathbb{E}\epsilon_i\epsilon_j = 0$, $cov(\epsilon_i, F) = 0$, $i = 1, \ldots, N$, $j \neq i$($\epsilon_i$ is called specific risk).

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# Arbitrage argument

Subtracting the original and the “expected” version of the “factor equation” we get that

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Its return:

$$R^p = \pi r = \sum_{i=1}^{k} \pi_i \mathbb{E}r_i + \sum_{i=1}^{k} \pi_i \epsilon_i \doteq \sum_{i=1}^{k} \pi_i \mathbb{E}r_i$$

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because $\pi$ is zero-financed and (nearly) riskless (otherwise an arbitrage would be possible)

# APT - continued

It follows from the assumptions of APT and from the arbitrage condition that there exist $\lambda_0, \ldots, \lambda_k$ such that

$$\mathbb{E}r_i = \lambda_0 + b_1\lambda_1 + \cdots + \beta_k\lambda_k$$

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Denote $r^z$ the return of the portfolio with $b_j = 0$, $j = 1, \ldots, k$. Clearly $\mathbb{E}r^z = \lambda_0$. (If a risk free rate $r_0$ exists then $\mathbb{E}r^z = r_0$ - it is because the risk free asset has to have zero $b$'s.)

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Further, if a portfolio has $b_j = 1$ and $b_\mu = 0$, $\mu \neq j$, then for its return $r^j$ it holds that $\mathbb{E} r^j = \lambda_0 + \lambda_1$, i.e. $\lambda_j = \mathbb{E} r^j - \mathbb{E} r^z$

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Therefore, the risk premium may be expressed as

$$\mathbb{E}r_i = \mathbb{E}r^z + b_{i,1}\mathbb{E}(r^1 - r^z) + \cdots + b_{i,k}\mathbb{E}(r^k - r^z)$$

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It follows from the assumptions of APT and from the arbitrage condition that there exist $\lambda_0, \ldots, \lambda_k$ such that

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Denote $r^z$ the return of the portfolio with $b_j = 0$, $j = 1, \ldots, k$. Clearly $\mathbb{E}r^z = \lambda_0$. (If a risk free rate $r_0$ exists then $\mathbb{E}r^z = r_0$ - it is because the risk free asset has to have zero $b$'s.)
Further, if a portfolio has $b_j = 1$ and $b_\mu = 0$, $\mu \neq j$, then for its return $r^j$ it holds that $\mathbb{E}r^j = \lambda_0 + \lambda_1$, i.e. $\lambda_j = \mathbb{E}r^j - \mathbb{E}r^z$

Therefore, the risk premium may be expressed as

$$\mathbb{E}r_i = \mathbb{E}r^z + b_{i,1}\mathbb{E}(r^1 - r^z) + \cdots + b_{i,k}\mathbb{E}(r^k - r^z)$$

Remark - the return of the marked portfolio may be one of the factors i.e. CAPM is a “special case” of APT.

# Homework (alternative)

Choose a single Czech stock and try to find its significant APT factors (e.g. foreign indices, exchange rate, whatever). The estimate should be based on at least 100 observations. The work is due by the midterm and should be sent to smid@utia.cas.cz.

# Multi-period models

[Samuelson 1969 (Nobel Prize 1970), Hakansson 1970]

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$$\max_{c_0,\ldots,c_T,\pi_0,\ldots,\pi_T} \mathbb{E}\left[\sum_{t=0}^{T} \rho^{-i} u_1(c_i) + u_2(w_T)\right]$$

$$w_t = \mathbf{1}'\pi_t + c_t, \quad w_{t+1} = r_t'\pi_t, \quad c_t, w_t \geq 0,$$
$$c_t, \pi_t \in \sigma(r_0, \ldots, r_{t-1}), \quad t = 0, \ldots, T-1$$

$w_0 \geq 0$ - initial wealth, $u_{1,2}$ - utility functions,
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## Theorem

*$u_2 \equiv 0$, $T = \infty$, $u_1$ is of a special type (CARA, CRRA), $r_{t,0}$ is risk free, $r_0, \ldots, r_T$ independent not a.s. above $r_{t,0}$ ...*

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# Notes on Solution of the Multi-period Models

1. Via nested reformulation (Bellmann's principle)
   - When we are lucky, we obtain a closed form or simple solution by a backwards solution.
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$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{N} \sigma_{i,j} dW_i(t)\right], \quad i = 1, \ldots, N,$$

$W(t)$ is the $N$-dimensional Wiener process

$r$, $b$ and $\sigma$ are "well behaved" functions.

# The Optimization Problem

$$\sup_{c,\pi} \mathbb{E}\left[\int_0^T u_1(c(t),t)dt + u_2(w(T))\right]$$

$$c \geq 0, \pi \in \mathbb{R}^N \quad \text{- adapted well behaved}$$

$$dw(t) = \pi(t)dS(t) + [(w(t) - \mathbf{1}'\pi(t))r(t) - c(t)]dt$$

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**Theorem**

*Under some regularity conditions, the solution is closed form and the proportions of risky assets in $\pi(t)$ do not depend on $u_{1,2}$.*

# Arbitrage Pricing Theory

$N$ stocks with vector of (stochastic) returns $r$

Assp. i. $r_i = a_{i,1} + b_{i,1}F_1 + \cdots + b_{i,k}F_K + \epsilon_i$ for some (stochastic) factors $F_1, \ldots, F_K$ where

- $\mathbb{E}\epsilon_i = 0$, $1 \leq i \leq N$,
- $\mathbb{E}\epsilon_i \epsilon j = 0$, $i \neq j$,
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Assp. (ii) implies $r_i = Er_i + b_{i,1}(F_1 - \mathbb{E}F_1) + \cdots + b_{i,k}(F_K - \mathbb{E}F_k) + \epsilon_i$

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Arbitrage argument: If $x_1, \ldots, x_N$ be such that

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Linear algebra $\Rightarrow \mathbb{E}r_i = \lambda_0 + \sum_{i=1}^{k} \lambda_k b_{i,k}$ for some $\lambda_0, \ldots, \lambda_k$.

# Drawbacks of the stock price models

- Returns are not normal
- Statistical properties of price processes contradicting standard models: autocorrelation of trade signs, autocorrelation of absolute returns. . .

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  $\Rightarrow$ *attempts to incorporate the properties into the models*

# Structural issues

- Agents have asymmetric information [Stiglitz, Nobel Prize 2001]
- Microstructure issues (transaction costs, bid-ask spread, little liquidity)

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  $\Rightarrow$ *e.g. my present work*

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- People do not optimize [Simon, Nobel Prize 1978]]
- People do not behave according to “standard” utility functions
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Radical solution: completely random behavior of the agents
(Farmer - still waiting for his Nobel Prize. . . )

# Related reading


- E. Barucci.
  *Financial Markets Theory*.
  Springer, London, 2003.
- J. Dupačová, J. Hurt, and J. Štěpán.
  *Stochastic Modelling in Economics and Finance*.
  Kluwer, Dodrecht, 2002.
- I. Karatzas and S. E. Shreve.
  *Methods of Mathematical Finance*.
  Springer, NY, 1998.
- R. Lowenstein.
  *When Genius Failed: The Rise and Fall of Long-Term Capital Management*.
  Random House, NY, Toronto, 2000.